

COUNTEREXAMPLES TO THE QUADRISECANT APPROXIMATION CONJECTURE

SHENG BAI, CHAO WANG, AND JIAJUN WANG

ABSTRACT. A quadrisecant of a knot is a straight line intersecting the knot at four points. If a knot has finitely many quadrisecants, one can replace each subarc between two adjacent secant points by the line segment between them to get the quadrisecant approximation of the original knot. It was conjectured that the quadrisecant approximation is always a knot with the same knot type as the original knot. We show that every knot type contains two knots, the quadrisecant approximation of one knot has self intersections while the quadrisecant approximation of the other knot is a knot with different knot type.

1. INTRODUCTION

A quadrisecant for a knot is a line that intersects the knot in four points. In 1933, Pannwitz [10] showed that a generic polygonal knot in any nontrivial knot type must have at least two quadrisecants. This result was extended to smooth knots by Morton and Mond [9] and to tame knots by Kuperberg [8]. Quadrisecants was used to give lower bounds of the ropelength of a knot in [4].

The quadrisecant approximation of a knot was introduced by Jin in [6]. For a knot K with finitely many quadrisecants, let W be the set of intersection points of K with the quadrisecants. The *quadrisecant approximation* of K , denoted by \widehat{K} , is obtained from K by replacing each subarc of K between two points in W adjacent along K with the straight line segment between them. If K has no quadrisecants (which has to be an unknot), we let $\widehat{K} = K$. A knot as the union of finitely many line segments is called a polygonal knot. It was showed in [6] that almost every polygonal knot has only finitely many quadrisecants (see also [1, 3]). In every knot type, there is a polygonal knot such that the quadrisecant approximation has the same knot type. In fact, the following conjecture was proposed in the same paper:

Conjecture 1.1 (The quadrisecant approximation conjecture). *If K has finitely many quadrisecants, then its quadrisecant approximation \widehat{K} has the same knot type as K . Furthermore K and \widehat{K} have the same set of quadrisecants.*

The conjecture were verified for some knots with crossing number not bigger than five in [6] and for hexagonal trefoil knots in [7].

In the present paper, we work on polygonal knots. For a polygonal knot, let $e(K)$ be the number of edges in K . For a knot type \mathcal{K} , the edge number $e(\mathcal{K})$ is the smallest number of edges among all polygonal knots with type \mathcal{K} . We show the following:

Theorem 1.2. *For any knot type \mathcal{K} , there exist polygonal knots K_* and K_\diamond of type \mathcal{K} such that the quadrisecant approximation \widehat{K}_* has self-intersections while \widehat{K}_\diamond is a knot with knot type different from \mathcal{K} .*

Furthermore, we can require that $e(K_) \leq e(\mathcal{K}) + 6$ and $e(K_\diamond) \leq \frac{5}{2}e(\mathcal{K}) + 17$. If \mathcal{K} does not contain the trefoil knot as a connected summand, we can require that $e(K_\diamond) \leq e(\mathcal{K}) + 14$.*

Hence Conjecture 1.1 does not hold in general. However, in view that our counterexamples contain redundant edges, and with the results in [7], the following weaker conjecture may still hold

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Conjecture 1.3. *For a polygonal knot K of type \mathcal{K} with $e(K)$ edges and finitely many quadrisecants, the quadrisecant approximation \widehat{K} has knot type \mathcal{K} .*

Given a knot type \mathcal{K} , let $R_*(\mathcal{K})$ be the minimal number $e(K) - e(\mathcal{K})$ for polygonal knots K with type \mathcal{K} such that \widehat{K} has self-intersections, and let $R_\diamond(\mathcal{K})$ be the minimal number $e(K) - e(\mathcal{K})$ for polygonal knots K with type \mathcal{K} such that \widehat{K} is a knot with type different from \mathcal{K} . Theorem 1.2 shows that $R_*(\mathcal{K}) \leq 6$ and $R_\diamond(\mathcal{K}) \leq \frac{3}{2}e(\mathcal{K}) + 17$ for general knots and $R_\diamond(\mathcal{K}) \leq 14$ for \mathcal{K} which does not contain the trefoil knot as a connected summand. Hence the following question is natural:

Question 1.4. *For a given knot type \mathcal{K} , what are $R_*(\mathcal{K})$ and $R_\diamond(\mathcal{K})$? In particular, are $R_*(\mathcal{K})$ and $R_\diamond(\mathcal{K})$ always positive?*

The second question is equivalent to Conjecture 1.3. On the other hand, in our examples, the knot type \widehat{K}_\diamond is the connected sum of K_\diamond with the trefoil knots. Hence we have the following question:

Question 1.5. *Given a knot type \mathcal{K} , what knot types can be given by \widehat{K} for knots K with type \mathcal{K} ? Is the knot type of \widehat{K} always a connected sum for which \mathcal{K} is a summand? Can \widehat{K} be in some sense “simpler” than K ? In particular, can the quadrisecant approximation \widehat{K} be the unknot for a knot K with nontrivial knot type?*

We remark that the examples in Theorem 1.2 can be modified to give corresponding examples in the smooth category, for example, by smoothing the corners.

The paper is organized as follows. In Section 2 we will explain how to find all quadrisecants of a given knot. In Section 3 we will give two trivial knots K_6 and K_{14} with 6 and 14 edges, such that \widehat{K}_6 has self-intersections while \widehat{K}_{14} is a trefoil knot. In Section 4 we will use a carefully defined connected sum operation to generalize the examples to show Theorem 1.2.

2. HOW TO FIND QUADRISECATS

In this section, we give an algorithm to find all quadrisecants of a given polygonal knot. The results will be used in the construction of counterexamples in the following sections.

Let $K = V_1V_2 \cdots V_n$ be a polygonal knot in \mathbb{R}^3 . The case when $n = 3$ is trivial, hence we always assume $n > 3$. Let $V_{n+1} = V_1$, and for $1 \leq i \leq n$, let $v_i = V_{i+1} - V_i$. We require that K is in general position, namely K satisfies following

- (a) no four vertices of K are coplanar;
- (b) the vectors for any three edges of K are linear independent.

Given three vectors u, v, w , let $\text{Det}(u, v, w)$ be their determinant. Then the above two conditions are equivalent to the following:

- (a') for $1 \leq i < j < k < l \leq n$,

$$\text{Det}(V_j, V_k, V_l) - \text{Det}(V_i, V_k, V_l) + \text{Det}(V_i, V_j, V_l) - \text{Det}(V_i, V_j, V_k) \neq 0;$$

- (b') for $1 \leq i < j < k \leq n$, $\text{Det}(v_i, v_j, v_k) \neq 0$.

If L is a quadrisecant of K , then there are four edges V_iV_{i+1} , V_jV_{j+1} , V_kV_{k+1} , V_lV_{l+1} , $1 \leq i < j < k < l \leq n$, such that for $h \in \{i, j, k, l\}$, we have

$$L \cap V_hV_{h+1} \in V_hV_{h+1} - \{V_{h+1}\},$$

namely the intersection point is in the interior of the edge or is the initial point of the edge. Then the intersection points can be presented as

$$(1) \quad V_i + pv_i, V_j + qv_j, V_k + rv_k, V_l + sv_l, 0 \leq p, q, r, s < 1.$$

Since the four points are collinear, there exist $x, y \in \mathbb{R} - \{0, 1\}$ such that

$$(2) \quad (1 - x)(V_i + pv_i) + x(V_j + qv_j) - (V_k + rv_k) = 0,$$

$$(3) \quad (1 - y)(V_i + pv_i) + y(V_j + qv_j) - (V_l + sv_l) = 0.$$

By (b') and the Cramer's Rule, we have

$$(4) \quad p = \frac{\text{Det}(V_k - V_j, v_j, v_k)}{\text{Det}(v_i, v_j, v_k)} \frac{1}{1-x} + \frac{\text{Det}(V_j - V_i, v_j, v_k)}{\text{Det}(v_i, v_j, v_k)},$$

$$(5) \quad q = \frac{\text{Det}(v_i, V_i - V_k, v_k)}{\text{Det}(v_i, v_j, v_k)} \frac{x-1}{x} + \frac{\text{Det}(v_i, V_k - V_j, v_k)}{\text{Det}(v_i, v_j, v_k)},$$

$$(6) \quad r = \frac{\text{Det}(v_i, v_j, V_j - V_i)}{\text{Det}(v_i, v_j, v_k)} \frac{x}{1} + \frac{\text{Det}(v_i, v_j, V_i - V_k)}{\text{Det}(v_i, v_j, v_k)},$$

$$(7) \quad p = \frac{\text{Det}(V_l - V_j, v_j, v_l)}{\text{Det}(v_i, v_j, v_l)} \frac{1}{1-y} + \frac{\text{Det}(V_j - V_i, v_j, v_l)}{\text{Det}(v_i, v_j, v_l)},$$

$$(8) \quad q = \frac{\text{Det}(v_i, V_i - V_l, v_l)}{\text{Det}(v_i, v_j, v_l)} \frac{y-1}{y} + \frac{\text{Det}(v_i, V_l - V_j, v_l)}{\text{Det}(v_i, v_j, v_l)},$$

$$(9) \quad s = \frac{\text{Det}(v_i, v_j, V_j - V_i)}{\text{Det}(v_i, v_j, v_l)} \frac{y}{1} + \frac{\text{Det}(v_i, v_j, V_i - V_l)}{\text{Det}(v_i, v_j, v_l)}.$$

Let $f(z) = 1/(1-z)$, then $f \circ f(z) = (z-1)/z$ and $f \circ f \circ f(z) = z$. Hence (4)(5)(6) and (7)(8)(9) are symmetric. Since $1 \leq i < j < k < l \leq n$, $V_i, V_{i+1}, V_k, V_{k+1}$ are four distinct points and $V_j, V_{j+1}, V_l, V_{l+1}$ are four distinct points. By (a'), we have

$$(10) \quad \text{Det}(v_i, V_i - V_k, v_k) \neq 0, \quad \text{Det}(V_l - V_j, v_j, v_l) \neq 0.$$

Hence by (4)(7) and (5)(8), x and y are dependent on each other. And we can solve x, y , then get p, q, r, s . Actually, x satisfies a quadratic equation

$$(11) \quad Ax^2 + Bx + C = 0.$$

Here A, B and C are some polynomial functions of coordinates of the vertices $V_i, V_{i+1}, V_j, V_{j+1}, V_k, V_{k+1}, V_l$ and V_{l+1} . And A or B or C may be zero.

In (4)(5)(6), let x vary in $\mathbb{R} - \{0, 1\}$, then we get a ruled surface S which is the union of lines passing through the corresponding $V_i + pv_i, V_j + qv_j$ and $V_k + rv_k$. If

$$\text{Det}(V_k - V_j, v_j, v_k) = 0, \quad \text{Det}(v_i, v_j, V_j - V_i) = 0,$$

then by (a'), we have $V_{j+1} = V_k$ and $V_{i+1} = V_j$. Then $p = 1, r = 0$, and L can not be a quadrisecant. If only one of the determinants is zero, then S is a part of a plane, and $V_i + pv_i$ or $V_k + rv_k$ is fixed when x varies. If both determinants are nonzero, then one can check that S is a quadric.

Hence if the equation (11) has infinitely many solutions, then $A = B = C = 0$, and the four edges $V_iV_{i+1}, V_jV_{j+1}, V_kV_{k+1}, V_lV_{l+1}$ must be linear dependent or lie on a quadric generated by V_iV_{i+1}, V_jV_{j+1} and V_kV_{k+1} . This is also the result in [6] that, for four edges in general position, there are at most two straight lines intersecting each of them, corresponding to two solutions of (11) with $A \neq 0$.

By the above discussion, for a given polygonal knot K , we have an algorithm to find all its quadrisecants, that is, to find all possible $0 \leq p, q, r, s < 1$ for every four edges of K . If K is in general position and no edge of K lies on a quadric generated by other three edges, then K have finitely many quadrisecants. We can then get its quadrisecant approximation. In the following section, we will use this algorithm to determine the quadrisecant approximation of a given polygonal knot. Most computations in the present paper are performed by Mathematica 6.0.

3. QUADRISECANT APPROXIMATION OF THE UNKNOT

In this section, we will give two polygonal unknots K_6 and K_{14} with 6 and 14 edges respectively such that $\widehat{K_6}$ has self-intersections, and $\widehat{K_{14}}$ is a trefoil knot.

3.1. Construction of K_6 . The 6-edge unknot K_6 is constructed as follows. Let $K = V_1V_2 \cdots V_6$ be the polygonal knot with the following coordinates:

$$\begin{aligned} V_1 &= (0, 0, 0), & V_2 &= (1, 0, 0), & V_3 &= (2, 0, 1), \\ V_4 &= (3, 0, 0), & V_5 &= (4, 0, 0), & V_6 &= (2, 3, 0). \end{aligned}$$

K is obtained from the triangle $V_1V_5V_6$ in Figure 1(a) by replacing a line segment in V_1V_5 by two edges V_2V_3 and V_3V_4 as in Figure 1(b). Clear V_1, V_2, V_4, V_5 are in a straight line ℓ . Then we can extend $V_6V_1, V_3V_2, V_3V_4, V_6V_5$ to get $K_6 = W_1W_2 \cdots W_6$ in Figure 1(c). With suitable choices of extensions, ℓ will be the only quadrisecant of K_6 , and \widehat{K}_6 will have self-intersections. The following is such a choice:

$$\begin{aligned} W_1 &= \left(-\frac{1}{5}, -\frac{3}{10}, 0\right), & W_2 &= \left(\frac{4}{5}, 0, -\frac{1}{5}\right), & W_3 &= (2, 0, 1), \\ W_4 &= \left(\frac{13}{4}, 0, -\frac{1}{4}\right), & W_5 &= \left(\frac{17}{4}, -\frac{3}{8}, 0\right), & W_6 &= (2, 3, 0). \end{aligned}$$

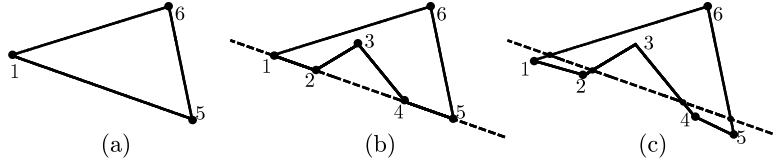


FIGURE 1. The hexagonal unknot K_6

3.2. Construction of K_{14} . We first construct a primary knot K_0 , then perturb it to get the final knot K_{14} . Certain lines intersecting the primary knot K_0 will become the quadrisecants of K_{14} .

The primary knot $K_0 = V_1V_2 \cdots V_{14}$ is the polygonal knot with the following coordinates, see Figure 2.

$$\begin{aligned} V_1 &= (0, 0, 0), & V_2 &= (0, 0, -4), & V_3 &= (8, -2, -4), \\ V_4 &= (6, -3, -6), & V_5 &= (0, 0, -6), & V_6 &= (0, 0, -8), \\ V_7 &= (10, -1, -8), & V_8 &= (10, -1, 1), & V_9 &= (6, 1, 0), \\ V_{10} &= (8, 0, -1), & V_{11} &= (6, -1, 0), & V_{12} &= (12, -2, -3), \\ V_{13} &= (12, 0, 0), & V_{14} &= (6, 2, 0). \end{aligned}$$

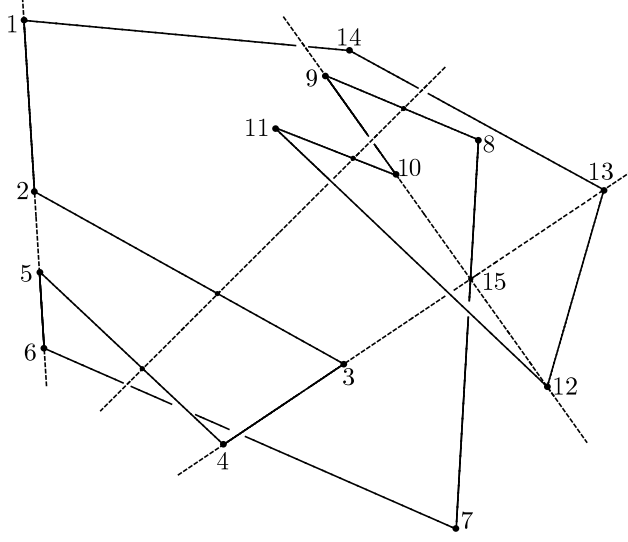
Let $V_{15} = (10, -1, -2)$. The knot K_0 is obtained as follows:

- (i) choose V_1, V_{13}, V_{14} and V_9 in the xy -plane, and V_2, V_5, V_6 in the z -axis;
- (ii) choose V_4 having the same z -coordinate as V_5 , and V_3, V_{15} in V_4V_{13} ;
- (iii) choose V_7V_8 containing V_{15} and parallel to the z -axis;
- (iv) choose V_{10}, V_{12} in the line passing through V_9 and V_{15} . Then choose V_{11} .

The following (a) and (b) can be verified:

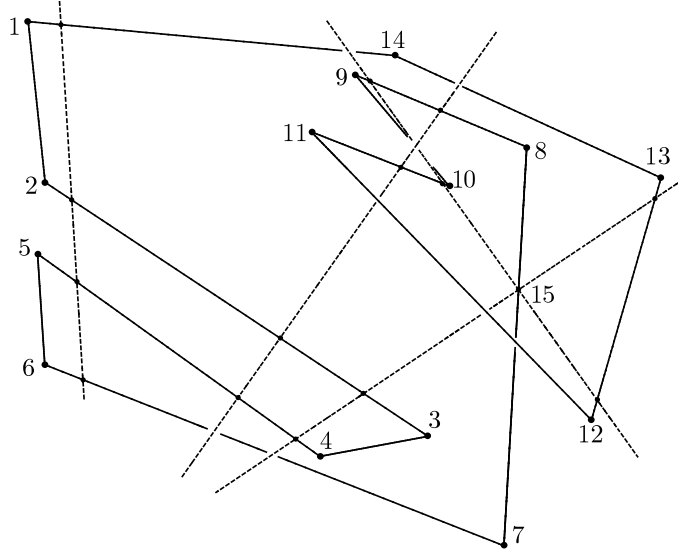
- (a) There are three straight lines L_1, L_2 and L_3 passing the four vertices in the three sets $\{V_1, V_2, V_5, V_6\}$, $\{V_3, V_4, V_{13}, V_{15}\}$ and $\{V_9, V_{10}, V_{12}, V_{15}\}$ separately;
- (b) There is a quadrisecant L_4 of K_0 intersecting V_2V_3, V_4V_5, V_8V_9 and $V_{10}V_{11}$.

In Figure 2, L_1, L_2, L_3, L_4 are given by the dashed lines. Let Λ be the broken line $V_1V_{14}V_{13}$. We hope that after small perturbation of the vertices, L_1, L_2, L_3 can become quadrisecants and no quadrisecant other than L_1 and L_2 can intersect Λ . By the discussion in Section 2, there will be a quadrisecant L'_4 intersecting V_2V_3, V_4V_5, V_8V_9 and $V_{10}V_{11}$. L'_4 can be thought as obtained from L_4 by a slightly movement. Then the quadrisecant approximation will be a trefoil knot.

FIGURE 2. The primary knot K_0

The final knot $K_{14} = W_1 W_2 \cdots W_{14}$ is given by the following coordinates, see Figure 3.

$$\begin{aligned}
 W_1 &= \left(-\frac{3}{5}, -\frac{1}{5}, 0\right), & W_2 &= \left(-\frac{19}{25}, \frac{11}{50}, -\frac{99}{25}\right), & W_3 &= \left(\frac{228}{25}, -\frac{66}{25}, -\frac{112}{25}\right), \\
 W_4 &= \left(\frac{143}{20}, -\frac{121}{40}, -\frac{109}{20}\right), & W_5 &= \left(-\frac{13}{10}, \frac{11}{20}, -\frac{61}{10}\right), & W_6 &= \left(-1, \frac{1}{10}, -\frac{171}{20}\right), \\
 W_7 &= (10, -1, -8), & W_8 &= (10, -1, 1), & W_9 &= \left(\frac{28}{5}, \frac{6}{5}, -\frac{1}{10}\right), \\
 W_{10} &= \left(\frac{81}{10}, \frac{1}{20}, -\frac{21}{20}\right), & W_{11} &= \left(\frac{59}{10}, -\frac{21}{20}, \frac{1}{20}\right), & W_{12} &= \left(12, -\frac{11}{5}, -\frac{33}{10}\right), \\
 W_{13} &= \left(12, \frac{1}{5}, \frac{3}{10}\right), & W_{14} &= (6, 2, 0).
 \end{aligned}$$

FIGURE 3. The final knot K_{14}

K_{14} is obtained from K_0 by firstly shrinking the edge V_3V_4 and slightly moving V_6 along z -axis, then extending edges V_7V_6 , V_4V_5 , V_2V_3 , $V_{14}V_1$, V_8V_9 , $V_{10}V_{11}$ and $V_{12}V_{13}$, according to the following

formulas.

$$\begin{aligned}
U_3 &= V_3 - (V_3 - V_4)\delta_3, & U_4 &= V_4 + (V_3 - V_4)\delta_4, \\
U_6 &= V_6 - (0, 0, \delta_6), & W_6 &= U_6 + (U_6 - V_7)\epsilon_6, \\
W_4 &= U_4 + (U_4 - V_5)\epsilon_4, & W_5 &= V_5 - (U_4 - V_5)\epsilon_5, \\
W_2 &= V_2 + (V_2 - U_3)\epsilon_2, & W_3 &= U_3 - (V_2 - U_3)\epsilon_3, \\
W_1 &= V_1 + (V_1 - V_{14})\epsilon_1, & W_9 &= V_9 + (V_9 - V_8)\epsilon_9, \\
W_{10} &= V_{10} + (V_{10} - V_{11})\epsilon_{10}, & W_{11} &= V_{11} - (V_{10} - V_{11})\epsilon_{11}, \\
W_{12} &= V_{12} + (V_{12} - V_{13})\epsilon_{12}, & W_{13} &= V_{13} - (V_{12} - V_{13})\epsilon_{13}.
\end{aligned}$$

Here the δ 's and ϵ 's are small positive numbers. The knot K_{14} is obtained by the following choices:

$$\begin{aligned}
\delta_3 &= \epsilon_3 = \epsilon_5 = 1/5, & \delta_4 &= 1/4, & \delta_6 &= 1/2, \\
\epsilon_1 &= \epsilon_2 = \epsilon_4 = \epsilon_6 = \epsilon_9 = \epsilon_{12} = \epsilon_{13} = 1/10, \\
\epsilon_{10} &= \epsilon_{11} = 1/20.
\end{aligned}$$

By computation using Mathematica, we found that K_{14} satisfies the general position conditions (a') and (b') in Section 2. By the discussion in Section 2 and Mathematica, it is not hard to see that K_{14} has only 4 quadriseccants, which are shown by the dashed lines in Figure 3.

K_{14} is a polygonal unknot. The quadriseccants and quadriseccant approximation of K_{14} are given in Figure 4. Its quadriseccant approximation \widehat{K}_{14} is a trefoil knot.

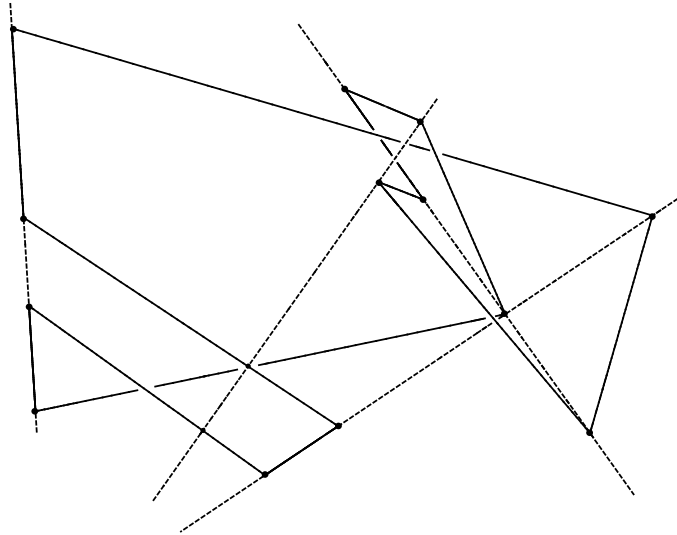


FIGURE 4. The quadriseccant approximation \widehat{K}_{14}

Remark 3.1. In the above example, K_{14} has two quadriseccants intersecting each other, which is not the generic case. We can perform slight perturbation to get disjoint quadriseccants while keeping all required results. For example, we can change V_{10} and V_{12} to $(8, 0, -1.1)$ and $(12, -2, -3.3)$ to achieve this, and the two corresponding intersection points on the edge V_7V_8 are $V_{15} = (10, -1, -2)$ and $V'_{15} = (10, -1, -2.2)$. We can also perturb V_3 to get disjoint quadriseccants.

4. QUADRISECCANT APPROXIMATION OF CONNECTED SUMS

In this section, we define a connected sum operation, and we use it to give counterexamples for general knot types. The counterexample will be a connected sum with one summand having a given knot type and satisfying certain conditions, and the other being the knot K_6 or K_{14} in Section 3.

4.1. The connected sum operation. Let $K = V_1 V_2 \cdots V_n$ be a polygonal knot in \mathbb{R}^3 . Given a plane Π in \mathbb{R}^3 , let K_Π be the image of K under the perpendicular projection from \mathbb{R}^3 onto Π . We can regard directed projections as points on the unit sphere $S^2 \subset \mathbb{R}^3$ with the induced topology.

Lemma 4.1 ([2, Proposition 1.12]). *For a given polygonal knot K , the set of projections whose image has only transverse double self-intersections is open and dense in S^2 .*

Suppose that Π is such a plane. We choose a vertex of the convex hull of K_Π , whose preimage must be some vertex, say V , of K .

Lemma 4.2. *Let K , Π and V be as above, and $\Sigma \subset \mathbb{R}^3$ be a plane perpendicular to the xy -plane. Then for any $P \in \Sigma$ and $\epsilon, \delta > 0$, there exists an affine transformation T from \mathbb{R}^3 to itself, such that:*

- (a) $T(\Pi)$ is parallel to the xy -plane.
- (b) $T(V) = P$ and $T(K)$ is in the ϵ -neighborhood of P .
- (c) $T(K) \cap \Sigma = P$, namely $T(K)$ lies in one side of Σ .
- (d) If a straight line L intersects more than two edges of $T(K)$, then the angle between L and $T(\Pi)$ is smaller than δ .

Proof. (a) can be achieved by rotations. (b) can be achieved by translations and linear contractions. Since the projection image of V in Π is a vertex of the convex hull of K_Π , (c) can be achieved by a further rotation around the line containing P and parallel to the z -axis. (a) and (b) will still hold. Then (d) can be achieved by a further linear contraction along the z -axis. \square

Note that condition (d) means that if the angle between a straight line L and the xy -plane is bigger than δ , then for a plane perpendicular to L the projection image of K in it will have only transverse double self-intersections.

Suppose that T is such an affine transformation. Denote $T(K)$ by $K_{\epsilon, \delta}^P$. One should keep in mind that the knot $K_{\epsilon, \delta}^P$ also depends on the plane Π , the vertex V , the plane Σ and the affine transformation T .

Lemma 4.3. *Let K and K' be two polygonal knots in \mathbb{R}^3 . Suppose that P is a point in K' , such that edges containing P are not parallel to the xy -plane. Then for sufficiently small $\epsilon, \delta > 0$, we have $K_{\epsilon, \delta}^P \cap K' = P$.*

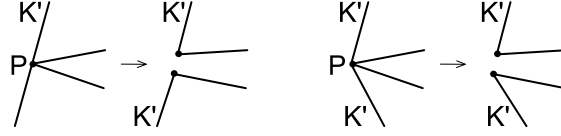
Proof. Let E be the union of edges of K' containing P . It will contain one or two edges. By (b) in Lemma 4.2, if ϵ is sufficiently small, then $K_{\epsilon, \delta}^P \cap K' \subset E$. Since edges in E are not parallel to the xy -plane, the angle between an edge in E and the xy -plane is nonzero. Since edges in E contain P , which belongs to two edges of $K_{\epsilon, \delta}^P$, by (a) and (d) in Lemma 4.2, if δ is sufficiently small, then E can not intersect $K_{\epsilon, \delta}^P - \{P\}$. Hence $K_{\epsilon, \delta}^P \cap K' = P$. \square

Let K, K', P, E be as in Lemma 4.3 and the proof. Denote the ϵ -neighborhood of P by $N_\epsilon(P)$. Then we can define the connected sum $K \#_P K'$ as following:

- (i) choose ϵ sufficiently small such that $N_\epsilon(P) \cap K' \subset E$;
- (ii) choose δ sufficiently small such that $K_{\epsilon, \delta}^P \cap K' = P$;
- (iii) let $K_{\epsilon, \delta}^P \vee K'$ be the one point union of $K_{\epsilon, \delta}^P$ and K' via P ;
- (iv) choose a way to resolve P in $K_{\epsilon, \delta}^P \vee K'$, and get a one component circle.

In (iv), P will be replaced by two points quite near P , see Figure 5 (In the left picture E contains one edge, and in the right picture E contains two edges. In each picture the arrow shows how to resolve P). Clearly the knot $K \#_P K'$ has the knot type of a connected sum of K and K' in the usual sense.

4.2. Counterexamples for general knots. Let K be a knot with n edges, and let Π, V be as above. When we choose $\Sigma, P, \epsilon, \delta$ and T in Lemma 4.2 suitably, we can have $K_{\epsilon, \delta}^P$. Let $K_{\epsilon, \delta}^P = V_1 V_2 \cdots V_n$ and $V_1 = P$. In following examples, P will be a vertex of K_6 or K_{14} , and we will consider the knots $K \#_P K_6$ and $K \#_P K_{14}$ which have the knot type of K .

FIGURE 5. From $K'_{\epsilon, \delta} \vee K'$ to $K \#_P K'$

In each connected sum the point P will be replaced by two points P^1 and P^2 , and $K'_{\epsilon, \delta}$ will become the broken line $P^1 V_2 \cdots V_n P^2$, which will be denoted by Ω . And the interior of a broken line M will be denoted by M° .

Proposition 4.4. *For any knot type \mathcal{K} there is a polygonal knot K_* of type \mathcal{K} with $e(\mathcal{K}) + 6$ edges such that \widehat{K}_* has self-intersections.*

Proof. In K_6 we choose $P = W_3 = (2, 0, 1)$, and let Σ be the plane containing W_2, W_3, W_4 . P will be replaced by the two points $W_3^1 = (2 - \eta, 0, 1)$ and $W_3^2 = (2 + \eta, 0, 1)$, here $\eta > 0$ is sufficiently small. Suppose that $P^1 = W_3^1$ and $P^2 = W_3^2$ such that the projection images of $P^1 V_2$ and $P^2 V_n$ in the xy -plane does not intersect. Then we can have the knot

$$K \#_P K_6 = W_1 W_2 P^1 V_2 \cdots V_n P^2 W_4 W_5 W_6.$$

Let Γ be the broken line $W_3^2 W_4 W_5 W_6 W_1 W_2 W_3^1$. It will have only one quadriseccant L . Let $L \cap W_1 W_6 = U_1$, $L \cap W_5 W_6 = U_2$, and let Λ be the broken line $U_1 W_6 U_2$.

By the discussion in Section 2, we can perturb the vertices of $K \#_P K_6$ slightly to get a knot K_* , such that K_* has finitely many quadriseccants. Ω, Γ, L and Λ will be moved slightly. If there is a quadriseccant L' of K_* intersecting Λ° , then it must intersect Ω° , since L is the only quadriseccant of Γ . Then, if the δ in Lemma 4.2 is sufficiently small, then $L' \cap \Omega^\circ$ can contain at most two points.

Case 1: $L' \cap \Omega^\circ$ contains two points. By (c) in Lemma 4.2, if the ϵ and δ are sufficiently small, then the y -coordinates of all such intersection points will have a positive lower bound, and a straight line passing such points and intersecting Λ° can intersect Σ at only one point. Then L' can not be a quadriseccant of K_* .

Case 2: $L' \cap \Omega^\circ$ contains one point. Then $L' \cap \Sigma$ contains three points, and L' must intersect $W_2 P^1$ and $W_4 P^2$. And it can not intersect Ω° , by (c) in Lemma 4.2.

The contradictions mean that no quadriseccant of K_* can intersect Λ° , and $U_1 U_2$ will be an edge of \widehat{K}_* . Then \widehat{K}_* will have self-intersections. \square

Proposition 4.5. *For any knot type \mathcal{K} , there is a polygonal knot K_\diamond of type \mathcal{K} with $e(\mathcal{K}) + 14$ edges, such that \widehat{K}_\diamond is a connected sum with the (left-handed) trefoil knot as a summand.*

Proof. In K_{14} we choose $P = W_7 = (10, -1, -8)$, and let Σ be the plane containing W_6, W_7, W_8 . P will be replaced by the two points $W_7^1 = (10 - 10\eta, \eta - 1, -8)$ and $W_7^2 = W_7$, here $\eta > 0$ is sufficiently small. Suppose $P^1 = W_7^1$ and $P^2 = W_7^2$ such that the projection images of $P^1 V_2$ and $P^2 V_n$ in the xy -plane does not intersect. Then we can have the knot

$$K \#_P K_{14} = W_1 W_2 \cdots W_6 P^1 V_2 \cdots V_n P^2 W_8 W_9 \cdots W_{14}.$$

Let Γ be the polygonal knot $W_7^2 W_8 W_9 \cdots W_{14} W_1 W_2 \cdots W_6 W_7^1$. It will have the same set of quadriseccants of K_{14} , namely L_1, L_2, L_3 and some L_4 described in Section 3. Let $L_1 \cap W_1 W_{14} = U_1$, $L_2 \cap W_{12} W_{13} = U_2$, and let Λ be the broken line $U_1 W_{14} W_{13} U_2$.

By the discussion in Section 2, we can perturb the vertices of $K \#_P K_{14}$ slightly to get a knot K_\diamond , such that K_\diamond has finitely many quadriseccants. Ω, Γ, Λ and the four quadriseccants L_1, L_2, L_3, L_4 will be moved slightly. If there is a quadriseccant L' of K_\diamond intersecting Λ° , then it must intersect Ω° , since no quadriseccant of Γ can intersect Λ° . Then, if the δ in Lemma 4.2 is sufficiently small, then $L' \cap \Omega^\circ$ contain at most two points.

Case 1: $L' \cap \Omega^\circ$ contains two points. By (c) in Lemma 4.2, if the ϵ and δ are sufficiently small, then a straight line passing such points and intersecting Λ° can intersect Σ at only one point. And L' can not be a quadrisecant of K_\diamond .

Case 2: $L' \cap \Omega^\circ$ contains one point. Then $L' \cap \Sigma$ contains three points, and L' must intersect $W_6 P^1$ and $W_8 P^2$. And it can not intersect Ω° , by (c) in Lemma 4.2.

The contradictions mean that no quadrisecant of K_\diamond can intersect Λ° , and $U_1 U_2$ will be an edge of \widehat{K}_\diamond . Since $\Omega \subset N_\epsilon(P)$, \widehat{K}_\diamond will be a connected sum of a trefoil knot and some knot K' . \square

If the knot K in the above proposition does not contain the trefoil knot as a connected summand (for example if K is a prime knot other than the trefoil knot), then \widehat{K}_\diamond will have the knot type different from K . For arbitrary knot type we have the following.

Proposition 4.6. *For any polygonal knot K with n edges, there is a polygonal knot K_\diamond^1 with the same knot type as K and $5\lfloor \frac{n+1}{2} \rfloor + 14$ edges, such that \widehat{K}_\diamond^1 has the knot type of a connected sum of K and a (left-handed) trefoil knot.*

Proof. The example of K_6 in Section 3 means that if we suitably replace a line segment in a chosen edge of K by two edges and suitably perturb the vertices, then there will be a quadrisecant L such that the union of line segments in L between the secant points is quite close to the chosen edge, as illustrated in Figure 6a. Here ‘quite close’ means that the Hausdorff distance between the two sets is

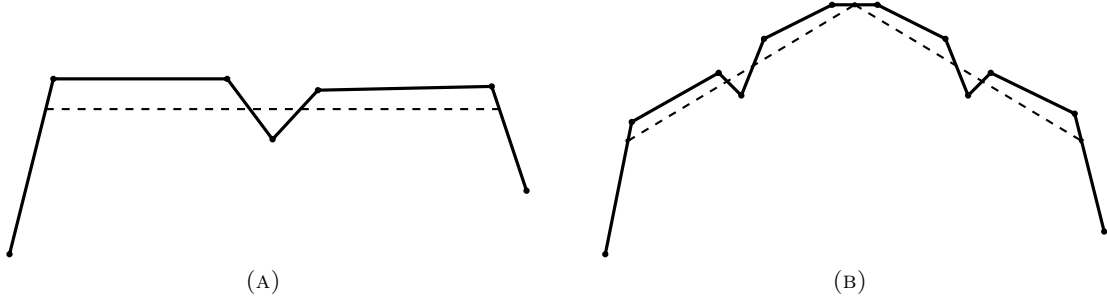


FIGURE 6. General knots. The dashed edge is the original edge, while the solid edges are the new edges.

sufficiently small.

We do this procedure successively for every other edge of K , such that each time the replacement happens in a sufficiently small neighborhood of the edge. When n is odd, we need change two adjacent edges and there will be one more edge at their common vertex, as in Figure 6b. We get a knot K^1 sufficiently close to K such that \widehat{K}^1 is also quite close to K . In particular, both K^1 and \widehat{K}^1 will have the same knot type of K .

K^1 has $5\lfloor \frac{n+1}{2} \rfloor$ edges. By the construction as in Proposition 4.5, we can get a K_\diamond^1 from K^1 . Since quadrisecants will be preserved under affine transformations, the Ω part will not change too much under the quadrisecant approximation. Then \widehat{K}_\diamond^1 has the knot type of a connected sum of K and the left-handed trefoil knot. \square

Proof of Theorem 1.2. It evidently follows from Propositions 4.4, 4.5 and 4.6. \square

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